RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 3 Fall 2023

1. Spherical polar coordinates

(a) The metric is a $(0, 2)$ tensor, and as such it transforms as follows

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g_{\alpha\beta}(x') \,. \tag{1}$$

Plugging in certain values of μ and ν, we retrieve the correct components. For example,

$$\begin{aligned} g_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 g_{xx} + \left(\frac{\partial y}{\partial r}\right)^2 g_{yy} + \left(\frac{\partial z}{\partial r}\right)^2 g_{zz} \\ &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1\,. \end{aligned} \tag{2}$$

This can be done for the other components, and you should eventually obtain

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\,. \tag{3}$$

(b) Given

$$x(\lambda) = \cos\lambda\,, \qquad y(\lambda) = \sin\lambda\,, \qquad z(\lambda) = \lambda\,, \tag{4}$$

we immediately derive

$$\begin{aligned} r(\lambda) &= \sqrt{x(\lambda)^2 + y(\lambda)^2 + z(\lambda)^2} = \sqrt{1+\lambda^2}\,, \\ \theta(\lambda) &= \arctan\left(\frac{x(\lambda)^2 + y(\lambda)^2}{z(\lambda)}\right) = \arctan\left(\frac{1}{\lambda}\right)\,, \\ \phi(\lambda) &= \arctan\left(\frac{y(\lambda)}{x(\lambda)}\right) = \lambda\,. \end{aligned} \tag{5}$$

(c) We use the decomposition of the derivative along the parameter λ in terms of derivatives in the coordinates

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu\,. \tag{6}$$

In the Cartesian coordinate system,

$$\frac{d}{d\lambda} = -\sin\lambda\partial_x + \cos\lambda\partial_y + \partial_z\,. \tag{7}$$

The components of the tangent vector in Cartesian coordinates are thus $(-\sin\lambda,\ \cos\lambda,\ 1)$. In the spherical polar coordinates,

$$\frac{d}{d\lambda} = \frac{\lambda}{\sqrt{1+\lambda^2}}\partial_r - \frac{1}{\lambda^2+1}\partial_\theta + \partial_\phi\,, \tag{8}$$

so that in spherical polar coordinates, the components of the tangent vector are $\left(\frac{\lambda}{\sqrt{1+\lambda^2}},\ -\frac{1}{\lambda^2+1},\ 1\right)$

2. Rindler coordinates

(a) You saw in the lectures that the dynamics of a particle in an electromagnetic field are described by

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = qF^{\alpha\beta}U_\beta\,, \tag{9}$$

where $U_\beta = \frac{\mathrm{d}x_\beta}{\mathrm{d}\tau}$ is the four-velocity. The spatial components give the familiar equation for the Lorentz force

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q(\vec{E} + \vec{v} \times \vec{B})\,, \tag{10}$$

where we used $\mathrm{d}t = \gamma \mathrm{d}\tau$. The time component describes the evolution of the particle's energy $\mathcal{E} \equiv p^0$

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = q\vec{E}\cdot\vec{v}\,. \tag{11}$$

We are interested in the case $\vec{B} = 0$ and $\vec{E} = (E, 0, 0)$ uniform. The equation (10) can be integrated. For a particle that starts at rest, we get

$$p_x(t) = qEt\,. \tag{12}$$

Moreover, equation (11) becomes

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = qE\frac{\mathrm{d}x}{\mathrm{d}t}\,, \tag{13}$$

which can be integrated to give

$$\mathcal{E}(t) = \mathcal{E}_0 + qE(x(t) - x_0)\,, \tag{14}$$

where $\mathcal{E}_0$ is the initial energy of the particle at rest. At the same time, we have that the mass shell condition reads

$$\mathcal{E}^2(t) = \mathcal{E}_0^2 + p^2(t) = \mathcal{E}_0^2 + (qEt)^2\,, \tag{15}$$

where $\mathcal{E}_0 \equiv mc^2$. That means that, necessarily,

$$x(t) = \frac{\mathcal{E}_0}{qE}\left[\sqrt{1 + \frac{q^2E^2}{\mathcal{E}_0^2}t^2} - 1\right] + x_0\,, \tag{16}$$

matching what is written in the problem set, when $a = \frac{qE}{\mathcal{E}_0}$. What happens for a massless particle?

(b) That can be done by setting $x_0 = \frac{1}{a}$. Then,

$$x = \frac{1}{a}\sqrt{1 + (at)^2} \tag{17}$$

and

$$x^2 - t^2 = \frac{1}{a^2} \tag{18}$$

which is verified by a parametrization

$$x = \frac{1}{a}\cosh\alpha\,, \qquad t = \frac{1}{a}\sinh\alpha\,, \tag{19}$$

for any α.

(c) With the parametrization discovered before, $x = \frac{1}{a}\cosh\alpha, t = \frac{1}{a}\sinh\alpha$ one can compute the interval along the path. Using the chain rule:

$$d\tau^2 = \mathrm{d}t^2 - \mathrm{d}x^2 = \left(\frac{1}{a}\cosh\alpha\right)^2 \mathrm{d}\alpha^2 - \left(\frac{1}{a}\sinh\alpha\right)^2 \mathrm{d}\alpha^2 = \frac{d\alpha^2}{a^2} \tag{20}$$

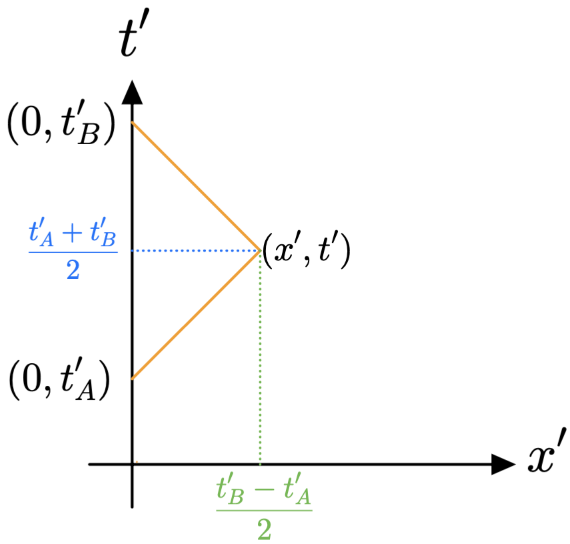
Proper time along this path is

$$\tau = \int \mathrm{d}\tau = \int \frac{1}{a}\mathrm{d}\alpha = \frac{\alpha}{a} \tag{21}$$

Inverting relation $t = \frac{1}{a}\sinh\alpha = \frac{1}{a}\sinh a\tau$ gives the relation described in the exercise,

$$\tau = \frac{1}{a}\sinh^{-1} at \tag{22}$$

(d) The rocket observer is measuring the time and position of P in terms of the time that elapses between the moment in which she sends the signal and the moment in which she receives it back. Since light moves at a constant speed, she defines t' as the average of the total time spent, and x' as half of the total distance traveled by the signal. That is the consistent way of building a reference frame for an observer in GR.



(e) Summing and subtracting her definitions of t' and x' we get

$$t'_A = t' - x'\,, \qquad t'_B = t' + x'\,. \tag{23}$$

From the laboratory observer instead we have

$$x - t = x_A - t_A\,, \qquad x + t = x_B + t_B\,. \tag{24}$$

For a generic motion $t = f_0(\tau)$ and $x = f_1(\tau)$ we have

$$\begin{aligned} t - x &= t_A - x_A = f_0(\tau_A) - f_1(\tau_A) = f_0(t' - x') - f_1(t' - x')\,, \\ t + x &= t_B + x_B = f_0(\tau_B) + f_1(\tau_B) = f_0(t' + x') + f_1(t' + x')\,, \end{aligned} \tag{25}$$

where we used that the proper time of the rocket is $\tau_A = t'_A$.

(f) In the specific case of uniform acceleration we can use the result from the first part of the exercise

$$f_0(\tau) = \frac{1}{a}\sinh(a\tau)\,, \qquad f_1(\tau) = \frac{1}{a}\cosh(a\tau)\,, \tag{26}$$

and obtain

$$\begin{aligned} t - x &= -\frac{1}{a}e^{-a(t'-x')}\,, \\ t + x &= \frac{1}{a}e^{a(t'+x')}\,, \end{aligned} \tag{27}$$

which leads to

$$\begin{aligned} t &= \frac{1}{a}\sinh(at')e^{ax'}\,, \\ x &= \frac{1}{a}\cosh(at')e^{ax'}\,. \end{aligned} \tag{28}$$

(g) We can check two things. First of all, we check that the time inside the rocket $(t'|_{x'=0})$ corresponds to what we called before proper time

$$t = \frac{1}{a}\sinh(at')e^{ax'} \quad \longrightarrow \quad \tau = t'|_{x'=0} = \frac{1}{a}\mathrm{arcsinh}(at)\,, \tag{29}$$

which exactly corresponds with the expression we had before. Moreover, we get

$$x|_{x'=0} = \frac{1}{a}\cosh(at') = \frac{1}{a}\sqrt{1+\sinh^2(at')} = \frac{1}{a}\sqrt{1+a^2t^2}\,, \tag{30}$$

again in correspondence to what we had before.

(h) From the transformation laws we derive that

$$\begin{aligned} \mathrm{d}t &= e^{ax'}\left[\cosh(at')\mathrm{d}t' + \sinh(at')\mathrm{d}x'\right]\,, \\ \mathrm{d}x &= e^{ax'}\left[\sinh(at')\mathrm{d}t' + \cosh(at')\mathrm{d}x'\right]\,, \end{aligned} \tag{31}$$

so

$$\mathrm{d}s^2 = e^{2ax'}\left(-\mathrm{d}t'^2 + \mathrm{d}x'^2\right)\,. \tag{32}$$

(i) Let us start from the transformation laws

$$\begin{aligned} t &= \frac{1}{a}\sinh(at')e^{ax'}\,, \\ x &= \frac{1}{a}\cosh(at')e^{ax'}\,. \end{aligned} \tag{33}$$

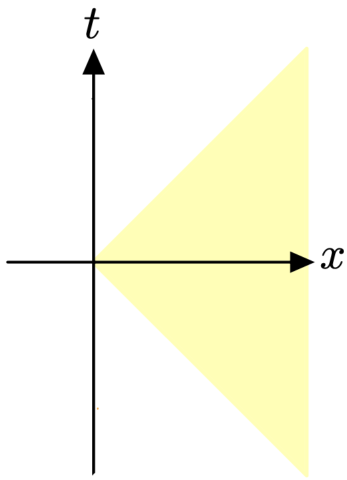
The Rindler coordinates are in the range

$$t' \in (-\infty,\infty) \qquad \text{and} \qquad x' \in (-\infty,\infty)\,, \tag{34}$$

Notice that

$$2\cosh(x) = e^x + e^{-x} > e^x - e^{-x} = 2\sinh(x) \tag{35}$$

so that necessarily $0 < t < x$. This is the Rindler patch. On the Minkowski plane, draw two lightrays, one going from the origin to $x = \infty$ and one going from $x = -\infty$ to the origin. The Rindler patch is the space in between these two lightrays.



(j) Consider the lightray going to the right, with some $x_0 < 0$

$$x(t) = x_0 + t \tag{36}$$

and the trajectory of the rocket (where we put $x_0 = \frac{1}{a}$ as suggested by the exercise)

$$x(t) = \frac{1}{a}\sqrt{1 + a^2t^2}\,. \tag{37}$$

We set them equal to each other to find whether there are intersections between the two trajectories. We square the two sides but we keep in mind that this means we will effectively be considering both $x(t) = \pm\frac{1}{a}\sqrt{1 + a^2t^2}$.

$$x_0^2 + t^2 + 2x_0t = \frac{1}{a^2} + t^2 \tag{38}$$

This has a solution

$$t_* = \frac{1}{2x_0a^2} - \frac{x_0}{2}\,. \tag{39}$$

We want to ask for which x_0 is this solution positive. The result is

$$x_0 < -\frac{1}{a}\,, \qquad 0 < x_0 < \frac{1}{a}. \tag{40}$$

This makes sense: the signal can reach the rocket only if it is sent at a distance between 0 and its initial position $\frac{1}{a}$. The solutions at $x_0 < -\frac{1}{a}$ are instead unphysical: they correspond with intersections between $x = x_0 + t$ and $x = -\frac{1}{a}\sqrt{1 + a^2t^2}$.

3. Surface

(a) To compute the length of a generic curve $\gamma : x^\mu(\lambda) = (r(\lambda), \phi(\lambda))$, we act with ds^2, a $(0, 2)$ tensor which takes two vectors as input and returns a number, on two copies of the tangent vector to the curve $V = \frac{dx^\mu}{d\lambda}\partial_\mu$.

$$ds^2(V, V) = g_{rr}(\lambda)\left(\frac{dr}{d\lambda}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2\,. \tag{41}$$

This is an infinitesimal distance squared. If we want to compute a finite distance over some values of λ, we take a square root and integrate over an interval

$$
\begin{aligned}
L_{ab} &= \int_a^b d\lambda \sqrt{g_{rr}(\lambda)\left(\frac{dr}{d\lambda}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2} \\
&= \int d\lambda \sqrt{\left(1+\frac{r^2(\lambda)}{a^2}\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2(\lambda)\left(\frac{d\phi}{d\lambda}\right)^2}
\end{aligned}
\tag{42}
$$

In this case we are interested in the curve $r(\lambda) = R$ and $\phi(\lambda) = \lambda$, on the interval $\lambda \in [0, 2\pi]$. This gives

$$
L_C = \int_0^{2\pi} R d\phi = 2\pi R\,. \tag{43}
$$

(b) To compute the area inside the curve, we use the fact that the correct invariant area element is given by $\epsilon = \sqrt{|g|}\mathrm{d}r \wedge \mathrm{d}\phi$. We get

$$
A_C = \int \sqrt{|g|} dr d\phi = 2\pi \int_0^R \sqrt{1+\frac{r^2}{a^2}} r dr \tag{44}
$$

we change variables to $\rho = \frac{r}{a}$ and we get

$$
A_C = \pi a^2 \int_0^{\frac{R}{a}} 2\rho\sqrt{1+\rho^2} d\rho = \frac{2}{3}\pi a^2 (1+\rho^2)^{\frac{3}{2}}\bigg|_0^{\frac{R}{a}} = \frac{2\pi a^2}{3}\left[\left(1+\frac{R^2}{a^2}\right)^{\frac{3}{2}} - 1\right]. \tag{45}
$$

(c) The distance from $r = 0$ to the curve C is given by

$$
L_R = \int_0^R \sqrt{g_{rr}} dr = \int_0^R \sqrt{1+\frac{r^2}{a^2}} dr\,. \tag{46}
$$

Given the hint in the exercise, we realize that the change of variables that will make our life simpler is $\frac{r}{a} = \sinh(x)$. The integral becomes

$$
L_R = a \int_0^{\sinh^{-1}\left(\frac{R}{a}\right)} \sqrt{1+\sinh^2 x} \cosh x \; dx = a \int_0^{\sinh^{-1}\left(\frac{R}{a}\right)} \cosh^2 x \; dx\,. \tag{47}
$$

Using the hint, we obtain

$$
\begin{aligned}
L_R &= a\left(\frac{x}{2} + \frac{1}{4}\sinh(2x)\right)\bigg|_0^{\sinh^{-1}\left(\frac{R}{a}\right)} \\
&= \frac{a}{2}\left(\sinh^{-1}\left(\frac{R}{a}\right) + \frac{R}{a}\cosh\left(\sinh^{-1}\left(\frac{R}{a}\right)\right)\right) \\
&= \frac{a}{2}\left(\sinh^{-1}\left(\frac{R}{a}\right) + \frac{R}{a}\sqrt{1+\frac{R^2}{a^2}}\right)
\end{aligned}
\tag{48}
$$

where we used $\sinh(2x) = 2\cosh(x)\sinh(x)$.

(d) A paraboloid is defined through the equation

$$z = c(x^2 + y^2) = cr^2 \, . \tag{49}$$

The metric in $\mathbb{R}^3$ is

$$\begin{aligned} ds^2 &= \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2 + \mathrm{d}z^2 \\ &= \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2 + (2cr\mathrm{d}r)^2 \\ &= \left(1 + 4c^2 r^2\right) \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2 \end{aligned} \tag{50}$$

To match with the given metric, we need $c = \frac{1}{2a}$, so that the paraboloid is

$$z = \frac{x^2 + y^2}{2a} \, . \tag{51}$$